A REMARK ON A THEOREM OF Y.KURATA.

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Abstract. In [K] Y.Kurata proved that the Goldie torsion theory splits centrally for dual rings. Here we extend his result to semilocal rings with left essential socle such that $\mathrm{Soc}\,({}_RR)^2 \subseteq \mathrm{Soc}\,(R_R)$. An example will demonstrate that our observation extends Kurata's result.

All rings are associative ring with unit and all R-modules are unital. The singular submodule of a left R-module M is denoted by $Z({}_RM)$. We abbreviate $S := \mathrm{Soc}\,({}_RR)$ and $J := \mathrm{Jac}\,(R)$ for the left socle resp. the Jacobson radical. We denote the left Goldie torsion theory, that is the hereditary torsion theory generated by all singular left R-modules, by τ_G and the torsion submodule of a module M by $\tau_G(M)$. τ_G is said to be centrally splitting if $\tau_G(R)$ is a ring direct summand of R. A classical result of Allin and Dickson [AD] states that τ_G is centrally splitting for a ring R if and only if R is a direct product of a semisimple ring and a ring with essential left singular ideal.

Lemma 1. *Let R be a ring with essential left socle S. Then*

(1) *$S = S^2 \oplus (S \cap Z({}_RR))$, where S^2 is projective and R/S^2 is τ_G-torsion.*
(2) *J is τ_G-torsion if and only if $S^2J = 0$.*

Proof. The socle can be decomposed as $S = S_0 \oplus S_1$ where $S_1 := S \cap Z({}_RR)$ and S_0 is a projective left R-module. $S^2 \subseteq S_0$, because for $x, y \in S$ with $x = x_0 + x_1$ and $y = y_0 + y_1$ where $x_0, y_0 \in S_0$ and $x_1, y_1 \in S_1$. The product $xy = xy_0 \in S_0$ as $xy_1 \in SZ({}_RR) = 0$. Thus $S^2 \subseteq S_0$ holds and there exists a left module $\tilde{S}$ such that $S_0 = S^2 \oplus \tilde{S}$. We have $S\tilde{S} \subseteq S^2 \cap \tilde{S} = 0$. If ${}_RS$ is essential in ${}_RR$, then $\tilde{S}$ becomes singular (as it is annihilated by S) and must be zero as it is also projective. Thus $S_0 = S^2$. Also R/S^2 becomes τ_G-torsion as $S_1 \simeq S/S^2$ and R/S are

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singular. This proves (1). Assume $S^2J = 0$, then J is an R/S^2-module and hence τ_G-torsion by (1). On the contrary, if J is τ_G-torsion, then $\mathrm{Hom}_R(S^2, J) = 0$ and therefore $S^2J = 0$. □

For a semilocal ring R we have $\mathrm{Soc}(R_R) = l.ann(\mathrm{Jac}(R))$, therefore the condition $S^2J = 0$ is equivalent to $\mathrm{Soc}({}_RR)^2 \subseteq \mathrm{Soc}(R_R)$.

Every semilocal ring R has a decomposition $R = R_0 \oplus R_1$ of left R-modules R_0 and R_1, where R_0 is semisimple artinian and J is essential in R_1. To see this, take a left ideal R_0 maximal with $R_0 \cap J = 0$. Then R_0 has a direct summand R_1 in R/J and it is easy to show that $R_0 \cap R_1 = 0$ as $R_0 \cap J = 0$.

Lemma 2. *Let R be a semilocal ring. Then τ_G is centrally splitting if and only if J has an essential singular left R-submodule. In this case* $\mathrm{Soc}({}_RR)^2 \subseteq \mathrm{Soc}(R_R)$ *holds.*

Proof. Note first, that J has an essential singular submodule if and only if J is τ_G-torsion. Recall the decomposition of semilocal rings $R = R_0 \oplus R_1$. " $\Rightarrow$ " Assume that every left R-module is a direct sum of a semisimple projective and a τ_G-torsion module. As R_1 has essential radical it has no simple direct summands, thus it must be τ_G-torsion and so must be $J \subseteq R_1$. " $\Leftarrow$ " as J is essential in R_1, R_1/J is singular. By hypothesis J is τ_G-torsion and so is R_1. Thus $\mathrm{Hom}_R(R_0, R_1) = 0 = \mathrm{Hom}_R(R_1, R_0)$ as R_0 is semisimple projective. Hence $R = R_0 \times R_1$ is a direct product of a semisimple ring and a ring with essential singular left submodule. By [AD] τ_G is centrally splitting.

As R_1 is τ_G-torsion it does not contain any projective simple submodule. Hence $S^2 = R_0$ and we have $S^2J = 0$ as $\mathrm{Hom}_R(R_0, R_1) = 0$. □

As a special case we get the following criterion for the splitting of τ_G for semilocal rings with essential left socle that extends Kurata's result for dual rings.

Theorem 3. *Let R be a semilocal ring with essential left socle. Then τ_G is centrally splitting if and only if* $\mathrm{Soc}({}_RR)^2 \subseteq \mathrm{Soc}(R_R)$.

Proof. The necessity is clear by Lemma 2. Assume that R is semilocal with essential left socle and $S^2J = 0$, then by Lemma 1(2) J is τ_G-torsion and by Lemma 2 the result follows. □

In order to verify that our result extends Kurata's result, we give an example of a commutative semilocal ring with essential simple socle that is not semiperfect and hence not a dual ring. I am very grateful to Patrick F. Smith for the following example.

Example (P.F.Smith)**.** *For any number n, there exists a commutative semilocal subdirectly irreducible non-local (and hence not semiperfect) ring with exactly n maximal ideals.* Take n different prime numbers $p_1, \ldots, p_n$. Then $R := \{\frac{a}{b} \in \mathbb{Q} \mid p_i \nmid b \ \forall i = 1, \ldots, n\}$ is a semilocal integral domain with n maximal ideals, which is not local. Let $M = \mathbb{Z}_{p_1^\infty}$ be the p_1-Prüfer group then M is a faithful R-module with essential simple socle isomorphic to $\mathbb{Z}/p_1\mathbb{Z}$. Form the trivial extension

$$S := R \propto M := \left\{ \begin{pmatrix} a & m \\ 0 & a \end{pmatrix} \mid a \in R, m \in M \right\}.$$

Then S is a commutative semilocal subdirectly irreducible non-local ring with exactly n maximal ideals.

Patrick Smith's results follows from the following lemma:

Lemma 4. Let R be a commutative semilocal ring, which is not local and assume there is a faithful subdirectly irreducible (SDI) R-module M. Then $S := R \propto M$ is a commutative semilocal SDI ring which is not local.

Proof. Let M be faithful with essential simple submodule N. Take an element $s = a \propto m \in S$. If $a \neq 0$, then $(a \propto m) \cdot (0 \propto M) = 0 \propto aM \neq 0$ since M is faithful. As $N \subseteq aM$ as R-modules we get $(0 \propto N) \subseteq (0 \propto aM) \subseteq sS$. If $a = 0$ and $m \neq 0$, then $sS = (0 \propto m)S = (0 \propto mR) \supseteq (0 \propto N)$ as $mR \supseteq N$. Thus S has an essential simple S-submodule $0 \propto N$. As $\mathrm{Jac}(S) = \mathrm{Jac}(R) \propto M$, S is semilocal, but not local as R is not local and S is indecomposable. □

References

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